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The chromatic number of the complementary graph to a simple graph associated with a commutative ring

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Abstract Let R be a Noetherian ring. We denote by $G(R)$ the simple graph whose vertices are elements of R and in which two distinct vertices x and y are joined by an edge if $x - y$ is a zero-divisor of R. Let $\overline{G}(R)$ be the complementary graph to $G(R)$ and $\overline{\chi}(R)$ be the chromatic number of the graph $\overline{G}(R)$. In this paper, we determine the chromatic number $\overline{\chi}(R)$. Let $P_1, \ldots, P_t$ be all maximal prime divisors of (0). If $\overline{\chi}(R)$ is finite, then

$$\overline{\chi}(R) = \min\{|R/P_i|; i = 1, 2, \ldots, t\}$$

where $|R/P_i|$ denotes the number of the set R/P_i.

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الملخص

لتكن R حلقة نوثيرية. نرمز بِـ $G(R)$ للبيان البسيط الذي تكون رؤوسه عناصر R ويتم فيه وصل رأسين مختلفين x وَ y بواسطة حافة إذا كان $x - y$ قاسماً للصفر في R. ليكن $\bar{G}(R)$ البيان المُتمِّم لِـ $G(R)$ وَ $\bar{\chi}(R)$ العدد اللَوْني للبيان $\bar{G}(R)$. نحدد في هذه الورقة العدد اللَوْني $\bar{\chi}(R)$. لتكن $P_1, \cdots, P_t$ جميع القواسِم الأولية الأعظمية لِـ (0). إذا كان $\bar{\chi}(R)$ منتهياً، فإن

$$\bar{\chi}(R) = \min\{|R/P_i|; i = 1, \cdots, t\}$$

حيث يرمز $|R/P_i|$ لعدد عناصر المجموعة R/P_i.

First, we collect the basic notions and results of graph theory for later use. For a simple graph G, $V(G)$ denotes the set of vertices of G and $E(G)$ denotes the set of edges of G. We color the vertices of G so that no two joined vertices have the same color. If we color the vertices, we call it a coloring of G. The chromatic number $\chi(G)$ of the graph G is the minimum number of colors needed to color G.

Let C be a non-empty subset of $V(G)$. We call C a clique of G if every pair of two distinct elements of C is joined by an edge. The clique number $C(G)$ of G is the number of vertices in the largest clique of G.

For a set S, $|S|$ denotes the number of element of S.

Our notation is standard and for unexplained terms, our general reference to commutative algebra is [1] and [5], and our general reference to graph theory is [2].

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Lemma 1 (cf. [6, Lemma 3]) *The inequality* $C(G) \leqq \chi(G)$ *holds.*

The symbol $\coprod$ denotes the disjoint union of sets.

Lemma 2 (cf. [6, Lemma 4]) *Let* $V_1, V_2, \ldots, V_t$ *be non-empty subsets of* $V(G)$. *Let*

$$V(G) = V_1 \coprod V_2 \coprod \cdots \coprod V_t.$$

be a disjoint union of $V(G)$ *such that no pair of two distinct elements of* V_i *is joined by an edge for* $i = 1, 2, \ldots, t$. *Then* $\chi(G) \leqq t$.

Let G_1 and G_2 be two simple graphs. We say that G_1 is a subgraph of G_2 if the following conditions hold: (1) $V(G_1) \subset V(G_2)$, (2) $E(G_1) \subset E(G_2)$.

The proof of the following lemma is obvious, so we omit it.

Lemma 3 *If* G_1 *is a subgraph of* G_2*, then* $\chi(G_1) \leqq \chi(G_2)$.

Let G be a simple graph. We define the complementary graph $\overline{G}$ of G to be a graph satisfying the following conditions:

(1) $V(\overline{G}) = V(G)$.
(2) Let x and y be two distinct vertices of $\overline{G}$. Then, x and y are joined by an edge in $\overline{G}$ if and only if x and y are not joined by an edge in G.

Let R be a commutative ring with the identity element. An element x is called a zero-divisor of R if there exists a non-zero element y of R such that $xy = 0$. We denote by $Z(R)$ the set of zero-divisors of R. We consider the simple graph $G(R)$ whose vertices are elements of R and in which two distinct vertices x and y are joined by an edge if $x - y$ is in $Z(R)$. Let $\chi(R)$ denote the chromatic number of the graph $G(R)$ and let $V(R)$ denote the set of vertices of $G(R)$.

$\overline{G}(R)$ denotes the complementary graph to $G(R)$ and $\overline{\chi}(R)$ denotes the chromatic number of the graph $\overline{G}(R)$.

In our previous paper [6], we have shown that if $\chi(R)$ is finite, then R is an integral domain or a finite ring. On the other hand, even if $\overline{\chi}(R)$ is finite, R is not necessarily a finite ring as the following example shows.

Before we proceed to Example 4, we collect some notions and results of commutative algebra for later use.

Let N be an R-module. For an element x of N, $\mathrm{Ann}_R(x) = \{c \in R;\, cx = 0\}$ is called the annihilator ideal of x. We say a prime ideal P of R is an associated prime ideal of N if there exists an element x of N such that $P = \mathrm{Ann}_R(x)$. $\mathrm{Ass}_R(N)$ denotes the set of all associated prime ideals of N. We consider the case $N = R$. If R is Noetherian, then $\mathrm{Ass}_R(R)$ consists of prime divisors of (0) in the primary decomposition of (0) ([5, (8.E) Lemma and (8.G) Theorem 11]).

Example 4 Let $\mathbf{Z}$ be the ring of integers and set $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$. Let $\mathbf{F}_2[S, T]$ be a polynomial ring over $\mathbf{F}_2$ in indeterminates S and T. Let s and t be the residue classes of S and T in $\mathbf{F}_2[S, T]/(ST, S^2)$ respectively. Then, $\mathbf{F}_2[s, t] = \mathbf{F}_2[S, T]/(ST, S^2)$. Set

$$R = \mathbf{F}_2[s, t][tX]$$

where X is an indeterminate and set $M = (s, t, tX)$.

Then the following assertions hold:

(1) M is a maximal ideal of R.
(2) $|R/M| = 2$.
(3) (s) is a prime ideal of R.
(4) $(0) = (s) \cap (t, tX)$.
(5) (t, tX) is an M-primary ideal of R.
(6) $\mathrm{Ass}_R(R) = \{(s), M\}$.
(7) $|R| = \infty$.
(8) $\overline{\chi}(R) < \infty$.

Proof (1) and (2) are trivial since R/M is isomorphic to $\mathbf{F}_2$.

(3) It is clear that $R/(s) \cong \mathbf{F}_2[T][TX]$ and $\mathbf{F}_2[T][TX]$ is an integral domain.

(4) Every element f of R is written as the following form:

$$a_0 + \sum_{n \geqq 1} a_n (tX)^n$$

where the sum $\sum_{n \geqq 1}$ is a finite sum, a_0 and a_n are in $\mathbf{F}_2[s, t]$ for $n = 1, 2, \ldots$. Furthermore, a_0 is written as

$$a_0 = b_0 + b_1 s + \sum_{m \geqq 1} c_m t^m$$

and $b_0, b_1, c_m \in \mathbf{F}_2$ for $m = 1, 2, \ldots$. Assume that f is in $(s) \cap (t, tX)$. Then $b_0 = b_1 = 0$ because $f \in (t, tX)$, and $a_n = c_m = 0$ for $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$ because $f \in (s) = \{0, s\}$. Hence $f = 0$. This implies that $(0) = (s) \cap (t, tX)$.

(5) Since $M^2 = (t^2, (tX)^2) \subset (t, tX) \subset M$ and M is maximal, (t, tX) is M-primary by [3, Ch. 4, section 2, no 1, example 3]).

(6) Let $\mathrm{Ann}_R(t)$ be the annihilator ideal of t. Then $(s) = \mathrm{Ann}_R(t)$. Similarly, we have $M = \mathrm{Ann}_R(s)$. Hence $\mathrm{Ass}_R(R) = \{(s), M\}$.

(7) Trivial.

(8) By Proposition 5 below, we see that $\overline{\chi}(R) < \infty$. □

We call Q a maximal prime divisor of (0) if Q is in $\mathrm{Ass}_R(R)$ and Q is a maximal element in $\mathrm{Ass}_R(R)$ with respect to inclusion. We know that $Z(R) = \cup Q$ where the union is taken over all maximal prime divisors Q's of (0) in $\mathrm{Ass}_R(R)$. ([4, Theorem 80 and its proof])

Proposition 5 *Assume that there is only one maximal element in* $\mathrm{Ass}_R(R)$*, say* M*. Then the following assertions hold:*

(1) $\overline{\chi}(R)$ *is finite if and only if* $|R/M| < \infty$.
(2) *If* $\overline{\chi}(R)$ *is finite, then* $\overline{\chi}(R) = |R/M|$.

Proof (1) Assume that $\overline{\chi}(R)$ is finite. Let $x_1 + M, x_2 + M, \ldots, x_t + M$ be distinct residue classes of R/M. Then $x_i - x_j \notin M = Z(R)$ for $i \neq j$. Therefore, $\{x_1, x_2, \ldots, x_t\}$ is a clique of $\overline{G}(R)$. By Lemma 1, $t \leqq \overline{\chi}(R)$ and hence $|R/M|$ is finite.
Conversely, assume that $|R/M| < \infty$. Set $|R/M| = t$ and let $x_1 + M, x_2 + M, \ldots, x_t + M$ be all residue classes of R/M. Note that $(x + a) - (x + b) = a - b \in M$ for elements $x + a, x + b$ in $x + M$. Set

$$V_1 = x_1 + M, V_2 = x_2 + M, \ldots, V_t = x_t + M.$$

Then, we see that

$$V(\overline{G}(R)) = V_1 \coprod V_2 \coprod \cdots \coprod V_t.$$

is a disjoint union of $V(\overline{G}(R))$ such that no pair of two distinct elements of V_i is joined by an edge for $i = 1, 2, \ldots, t$. Hence by Lemma 2, we get $\overline{\chi}(R) \leqq t$. This shows that $\overline{\chi}(R)$ is finite.
(2) From the argument of the proof of the assertion (1), it is easily seen that $\overline{\chi}(R) = |R/M|$. □

Lemma 6 *Let* $P_1, \ldots, P_t$ *be distinct prime ideals of* R*. Let* n *be a positive integer and assume that*

$$|R/P_1| \geqq n, \ldots, |R/P_t| \geqq n.$$

Then there exist elements $z_1, \ldots, z_n$ *of* R *such that*

$$z_i - z_j \notin P_1 \cup \cdots \cup P_t$$

for $i, j \in \{1, 2, \ldots, n\}$ *with* $i \neq j$.

Proof Let

$$x_u^{(1)} + P_u, x_u^{(2)} + P_u, \ldots, x_u^{(n)} + P_u$$

be distinct residue classes of R/P_u for $u = 1, 2, \ldots, t$. Set $S = R - P_1 \cup \cdots \cup P_t$ and $I = P_1 \cap \cdots \cap P_t$.Then R_S is a semi-local ring with maximal ideals $(P_1)_S, \ldots, (P_t)_S$. By Chinese remainder theorem we have an isomorphism

$$\phi : R_S/I_S \longrightarrow R_S/(P_1)_S \times \cdots \times R_S/(P_t)_S$$

by

$$\phi\left(\frac{x}{s} + I_S\right) = \left(\frac{x}{s} + (P_1)_S, \ldots, \frac{x}{s} + (P_t)_S\right).$$

Set

$$\begin{aligned}
\boldsymbol{\alpha}_1 &= (x_1^{(1)} + (P_1)_S, x_2^{(1)} + (P_2)_S, \ldots, x_t^{(1)} + (P_t)_S),\\
\ldots&\\
\boldsymbol{\alpha}_n &= (x_1^{(n)} + (P_1)_S, x_2^{(n)} + (P_2)_S, \ldots, x_t^{(n)} + (P_t)_S).
\end{aligned}$$

Then there exist elements $\frac{y_1}{s_1}, \ldots, \frac{y_n}{s_n}$ of R_S ; ($y_1, \ldots, y_n \in R$ and $s_1, \ldots, s_n \in S$) such that

$$\phi\left(\frac{y_1}{s_1} + I_S\right) = \alpha_1, \ldots, \phi\left(\frac{y_n}{s_n} + I_S\right) = \alpha_n.$$

Set $s = s_1 \cdot \cdots \cdot s_n$ and

$$z_1 = s\frac{y_1}{s_1}, \ldots, z_n = s\frac{y_n}{s_n}.$$

Then $z_1, \ldots, z_n$ are elements of R and

$$\phi(z_1 + I_S) = s\alpha_1, \ldots, \phi(z_n + I_S) = s\alpha_n.$$

We shall show that $z_i - z_j \notin P_1 \cup \cdots \cup P_t$ for $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. Suppose the contrary. Then there are indices i, j with $i \neq j$ and u such that $z_i - z_j \in P_u$. Note that $x_u^{(i)} - x_u^{(j)} \notin P_u$, $z_i - sx_u^{(i)} \in (P_u)_S$ and $z_j - sx_u^{(j)} \in (P_u)_S$. Then $z_i - z_j - (sx_u^{(i)} - sx_u^{(j)})$ is in $(P_u)_S$. Since $z_i - z_j \in P_u$, we get $s(x_u^{(i)} - x_u^{(j)}) \in (P_u)_S$. This implies that $x_u^{(i)} - x_u^{(j)} \in P_u$ because $s \notin P_u$. This is a contradiction. □

Proposition 7 *Let R be a Noetherian ring and let $P_1, \ldots, P_t$ be all maximal prime divisors of* (0). *If $\overline{\chi}(R)$ is finite, then there exists an index u such that $|R/P_u|$ is finite.*

Proof Assume that $\overline{\chi}(R)$ is finite and

$$|R/P_1| = \infty, \ldots, |R/P_t| = \infty.$$

Set $n = \overline{\chi}(R) + 1$. Then by Lemma 6 there exist elements $z_1, \ldots, z_n$ of R such that

$$z_i - z_j \notin P_1 \cup \cdots \cup P_t = Z(R)$$

for $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. Hence $\{z_1, \ldots, z_n\}$ is a clique of $\overline{G}(R)$, and by Lemma 1 we have $n \leqq \overline{\chi}(R) = n - 1$. This is absurd. Therefore, there exists an index u such that $|R/P_u|$ is finite. □

Lemma 8 *Let P be a prime ideal of R. Let G' be the graph whose vertices are elements of R and in which two distinct vertices x and y are joined by an edge if $x - y \notin P$. If $|R/P|$ is finite, then $\chi(G') = |R/P|$.*

Proof Set $n = |R/P|$ and let $x_1+P, \ldots, x_n+P$ be distinct residue classes of R/P. Set $V_1 = x_1+P, \ldots, V_n = x_n + P$. Then

$$V(G^{'}) = V_1 \coprod \cdots \coprod V_n$$

is a disjoint union of $V(G^{'})$ such that no pair of two distinct elements of V_i is joined by an edge for $i = 1, 2, \ldots, n$. By Lemma 2, we get $\chi(G^{'}) \leqq n$. On the other hand, $\{x_1, \ldots, x_n\}$ is a clique of $G^{'}$, hence $\chi(G^{'}) \geqq n$. Therefore, $\chi(G^{'}) = |R/P|$. □

Lemma 9 *Let R be a Noetherian ring and let P be a maximal prime divisor of* (0). *If $|R/P|$ is finite, then $\overline{\chi}(R)$ is finite and $\overline{\chi}(R) \leqq |R/P|$.*

Proof Let $G^{'}$ be the graph as in Lemma 8. Since $P \subset Z(R)$, we see that $\overline{G}(R)$ is a subgraph of $G^{'}$. Lemma 3 implies that $\overline{\chi}(R) \leqq \chi(G^{'})$. By Lemma 8, we have $\chi(G^{'}) = |R/P|$. Hence, $\overline{\chi}(R)$ is finite and $\overline{\chi}(R) \leqq |R/P|$. □

Theorem 10 *Let R be a Noetherian ring. Then the following assertions hold:*

(1) *$\overline{\chi}(R)$ is finite if and only if there exists an element P of* $\mathrm{Ass}_R(R)$ *such that $|R/P|$ is finite.*
(2) *Let $P_1, \ldots, P_t$ be all maximal prime divisors of* (0). *If $\overline{\chi}(R)$ is finite, then*

$$\overline{\chi}(R) = \min\{|R/P_i|; i = 1, 2, \ldots, t\}.$$

Proof (1) If $\overline{\chi}(R)$ is finite, then by Proposition 7 there exists a maximal prime divisor P of (0) such that $|R/P|$ is finite.
Conversely, if there exists an element P of $\mathrm{Ass}_R(R)$ such that $|R/P|$ is finite. Then there exists a maximal prime divisor Q of (0) such that $P \subset Q$. Since $|R/Q| \leqq |R/P|$, we see that $|R/Q|$ is finite. Then, Lemma 9 shows that $\overline{\chi}(R)$ is finite.

(2) If $\overline{\chi}(R)$ is finite, then $\min\{|R/P_i|; i = 1, 2, \ldots, t\}$ is finite by the argument of the proof of the assertion (1). Set $n = \min\{|R/P_i|; i = 1, 2, \ldots, t\}$. Then $|R/P_1| \geqq n, \ldots, |R/P_t| \geqq n$. By Lemma 6, there exist elements $z_1, \ldots, z_n$ of R such that $z_i - z_j \notin P_1 \cup \cdots \cup P_t = Z(R)$ for $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. Hence $\{z_1, \ldots, z_n\}$ is a clique of $\overline{G}(R)$ and $\overline{\chi}(R) \geqq n$ by Lemma 1. On the other hand, by Lemma 9, $\overline{\chi}(R) \leqq n$. Hence

$$\overline{\chi}(R) = n = \min\{|R/P_i|; i = 1, 2, \ldots, t\}.$$

□

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